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Feedback control of the two-phase Stefan problem, with an application to the continuous casting of steel

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***Abstract*—A full-state feedback control law is derived that stabilizes the two-phase Stefan problem with respect to a reference solution using control of the Neumann boundary condition. Stability and convergence are shown via a Lyapunov functional on the error system with moving boundaries. A second control law is also derived, for which stability is proved and convergence is conjectured due to the clearly convergent simulation results. A simple Dirichlet controller is also considered, and is used to design a boundary-output-based estimator that, in combination with full-state feedback controllers, yields a plausible output feedback control law with boundary sensing and actuation. The performance of the control laws is demonstrated using numerical simulation.**

I. INTRODUCTION

THE temperature of a solidifying, pure material can be described by a non-linear partial differential equation (PDE) of the form commonly known as a Stefan problem. This problem divides the domain into two or more time-varying subdomains separated by moving boundaries. In a model of an industrial casting process, these domains correspond to the solid and liquid phases of the material. The movement of the boundary between the phases is described by the Stefan condition, a differential equation derived from an energy balance at the boundary that is a function of the left and the right spatial derivatives of the temperature at the boundary[1].

The Stefan problem can be used to model a variety of industrial casting applications. The present paper focuses on the continuous casting of steel. Key quality and safety goals for this process can be achieved by matching an ideal temperature history. For example, certain cracks can be prevented by keeping the temperature in a specific range, since the ductility of steel is a function of temperature and time. In a continuous caster, the steel is contained within rolls that drive the steel downwards. If the steel is not fully solidified when it exits the last of these rolls, a severe bulge, known as a "whale," is formed due to the pressure of the liquid steel. This can be prevented by ensuring that the temperature through the entire cross-section is below the solidification temperature before the end of containment.

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Since these goals can be achieved via temperature regulation, an automated control strategy based on the steel temperature would be greatly beneficial to the industry. However, most casters use open-loop methods to control the flow rates of the water sprays that cool the surface. A key obstacle to the design of such control is the strong inherent nonlinearity of the solidification evolution equations. The many results in recent years for control of linear or even semi-linear distributed parameter systems are not applicable.

Previous work on control of processes governed by the Stefan problem can be generally divided into three categories: numerical optimization methods [2, 3], solutions of the inverse Stefan problem [4-6], and feedback control methods [7-10]. The numerical optimization methods in [2] and [3] can take into account realistic metallurgical constraints and quality conditions. However, since the simulation involved is highly complex and nonlinear, they cannot realistically run in real-time. The inverse methods and feedback control methods, with the exception of [10], use the Stefan problem as a model and focus on control of the boundary position, which would ensure whale prevention, but not necessarily the steel quality. The inverse problem, as solved in [4] and [5] directly and in [6] by minimizing a cost functional, is very numerically complex and thus limited to design of open-loop controllers. The feedback control methods are better suited for real-time control, but the control in [7] and [8] is simplified to the "on-off" thermostat-style one. In [9] and [10], PI controllers are designed based on a discretized form of the solidification evolution equations. However, neither controls the full temperature distribution. [9] only considers the solidification boundary, while [10] focuses on the steel surface temperature.

Our goal in this paper is to stabilize the solution to the Stefan problem relative to a reference solution. The latter is assumed to be safe and provide good metallurgical quality under nominal conditions, so that the process goals are met by reducing the reference error to zero. Our key tool is a Lyapunov functional on solutions of the Stefan problem with a moving boundary. This allows for the construction of a control law that stabilizes the error, and shows convergence of the error to zero asymptotically.

In Section II, we give a brief description of the problem. In Section III, we provide two control laws for the Neumann boundary condition of the Stefan problem and the associated proof of convergence for one of them. In Section IV, we examine the relevance of these assumptions for the application considered. To address controller implementability, in Section V we offer a simple Dirichlet control law, which allows for the design of an estimator based on boundary temperature measurements, that, when combined with controllers of Section III, yields a plausible output feedback controller with boundary actuation and sensing (a useful background for identification and control with the latter features for fixed boundary problems is given in [11] and [12]). In Section VI, we give simulation results that support the theorems and conjectures made in the paper.

II. The Two-Phase Stefan Problem

A. Problem Description

The solidifying steel in a continuous slab caster, called the strand, is rectangular. The outer solid shell of the strand encloses the liquid core. The relative size of the latter decreases as the strand is cooled, giving rise to the internal moving solid/liquid interface. The temperature evolution equations for the strand are three dimensional, and must account for, at the least, heat diffusion, advection at the casting speed, and the phase change. However, through symmetry and scaling arguments, the problem reduces to a one-dimensional "slice" that moves through the caster at the casting speed. A more detailed discussion of this modeling setting can be found in [13].

We denote the temperature within the slice as $T(x,t)$, for $0 \le x \le L$ and $t \ge 0$, where $x=0$ and $x=L$ correspond to the outer surface and the center of the strand, respectively. The boundary between solid and liquid phases is denoted $s(t)$. Then the following partial differential equation (PDE) models the evolution of temperature within the slice:

$$T_t(x,t) = aT_{xx}(x,t),\ 0<x<s(t),\ s(t)<x<L,$$
$$T(s(t),t) = T_f, \tag{1}$$

$$T_x(0,t) = u(t),\quad T_x(L,t)=0,\ T(x,0)=T_0(x),$$
$$\dot{s}(t) = b\left(T_x\left(s^-(t),t\right) - T_x\left(s^+(t),t\right)\right),\ s(0)=s_0. \tag{2}$$

In physical terms, T_f is the melting temperature, a is the thermal diffusivity, and $b = k/\rho L_f$, where k is the thermal conductivity, ρ is the density, and L_f is the latent heat of fusion. These physical quantities are all strictly positive. The control input $u(t)$ is applied as the Neumann boundary condition at $x=0$. In the caster, this is directly proportional to the heat flux removed from the steel at the surface.

For the convergence proof, we will need the following assumptions on the initial conditions:

(A1) $0 < s_0 < L$ and $T_0(s_0) = T_f$, $T(x,t) < T_f$ for $0 \le x < s(t)$, and $T(x,t) \ge T_f$ for $s(t) \le x \le L$,

(A2) $T_0(x)$ is continuous on $[0,L]$ and infinitely differentiable except at s_0.

The assumptions, respectively, ensure that the equations are well defined at $t=0$ and that solutions have sufficient regularity. Throughout this paper, we deal with the case in which $\varepsilon < s(t) < L-\varepsilon$ for some $\varepsilon > 0$, that is when the slice is neither fully solid nor liquid and the Stefan problem is well defined. If this is not true, the problem is linear and known distributed parameter control methods, e.g. those in [12], may be used.

B. Reference System and Error

We assume that we have a known reference temperature $\bar{T}(x,t)$ and solidification front position $\bar{s}(t)$, that are the solutions to (1)-(2) under known reference control input $\bar{u}(t)$ with initial conditions $\bar{T}(x,0) = \bar{T}_0(x)$ and $\bar{s}(0) = \bar{s}_0$ satisfying assumptions (A1) and (A2). This reference temperature profile should satisfy the metallurgical goals and constraints of the process, and could, for example, be calculated for the continuous caster using previous results, e.g [2-6]. That is, matching the reference temperature should result in safe operation and good quality steel. We add one more assumption on the reference profile:

(A3) $\dot{\bar{s}}(t) > 0$ for all $t \ge 0$.

We denote the errors as $\tilde{T}(x,t) = T(x,t) - \bar{T}(x,t)$, and $\tilde{s}(t) = s(t) - \bar{s}(t)$. Also, we denote $\tilde{u}(t) = u(t) - \bar{u}(t)$. Subtracting the PDEs yields

$$\tilde{T}_t(x,t) = a\tilde{T}_{xx}(x,t),\ x \in (0,L)\setminus\{s,\bar{s}\}. \tag{3}$$

Also, since solutions to (1)-(2) are twice spatially differentiable outside of the solidification front, they must have continuous first spatial derivatives. Thus, if $\bar{s}(t) \ne s(t)$, then $\bar{T}_x\left(s^+(t),t\right) = \bar{T}_x\left(s^-(t),t\right)$, and so

$$\dot{s}(t) = b\left(\tilde{T}_x\left(s^-(t),t\right) - \tilde{T}_x\left(s^+(t),t\right)\right). \tag{4}$$

Similarly,

$$\dot{\bar{s}}(t) = -b\left(\tilde{T}_x\left(\bar{s}^-(t),t\right) - \tilde{T}_x\left(\bar{s}^+(t),t\right)\right). \tag{5}$$

In the remainder of the paper, we will employ a simplified notation, using $T(x)$ to represent $T(x,t)$, or omitting both arguments altogether.

III. Control Law

The main result of this paper is stated as follows:

Theorem 1. Let the system (1)-(2) be controlled such that

$$u(t) = \bar{u}(t) - \frac{1}{\tilde{T}(0)+\tilde{T}_{xx}(0)}\Bigg[\tilde{T}_x(x)\left(\tilde{T}(x)+\tilde{T}_{xx}(x)\right)\Big|_{s^-}^{s^+} + \tilde{T}_x(x)\left(\tilde{T}(x)+\tilde{T}_{xx}(x)\right)\Big|_{\bar{s}^-}^{\bar{s}^+} + \frac{1}{2a}\dot{s}(t)\tilde{T}_x^2(x)\Big|_{s^-}^{s^+} + \frac{1}{2a}\dot{\bar{s}}(t)\tilde{T}_x^2(x)\Big|_{\bar{s}^-}^{\bar{s}^+}\Bigg] \tag{6}$$

where the initial conditions satisfy (A1) and (A2), and the reference solidification front position satisfies (A3). Then the reference error $\tilde{T}(x,t)$ converges uniformly to 0 as $t \to \infty$.

Proof: Consider the Lyapunov functional

$$V(\tilde{T}) := \frac{1}{2}\int_0^L \left(\tilde{T}^2 + \tilde{T}_x^2\right)dx = \frac{1}{2}\left[\int_0^{s_1}\left(\tilde{T}^2 + \tilde{T}_x^2\right)dx + \int_{s_1}^{s_2}\left(\tilde{T}^2 + \tilde{T}_x^2\right)dx + \int_{s_2}^{L}\left(\tilde{T}^2 + \tilde{T}_x^2\right)dx\right], \quad (7)$$

where $s_1 := \min\{s, \bar{s}\}$ and $s_2 := \max\{s, \bar{s}\}$. Note that $V(\tilde{T})$ is equivalent to the square of the Sobolev norm,

$$\left\|\tilde{T}\right\|_{1,2} := \left\|\tilde{T}\right\|_2 + \left\|\tilde{T}_x\right\|_2, \quad (8)$$

in the sense that

$$\frac{1}{2}\left\|\tilde{T}\right\|_{1,2}^2 \geq V(\tilde{T}) \geq \frac{1}{4}\left\|\tilde{T}\right\|_{1,2}^2. \quad (9)$$

Since solutions of the Stefan problem are twice differentiable except at the boundary, the first weak derivative exists and such solutions are in the Sobolev space $W^{1,2}(0,L)$.

Assuming that $\bar{s}(t) \neq s(t)$, and ignoring the degenerate case for now, taking the time derivative of (7) using the PDE (3) yields:

$$\dot{V}(\tilde{T},t) = -\frac{1}{2}\dot{s}_1\left(\tilde{T}^2 + \tilde{T}_x^2\right)\Big|_{s_1^-}^{s_1^+} - \frac{1}{2}\dot{s}_2\left(\tilde{T}^2 + \tilde{T}_x^2\right)\Big|_{s_2^-}^{s_2^+} + \int_0^{s_1}\left(\tilde{T}\tilde{T}_{xx} + \tilde{T}_x\tilde{T}_{xxx}\right)dx + \int_{s_1}^{s_2}\left(\tilde{T}\tilde{T}_{xx} + \tilde{T}_x\tilde{T}_{xxx}\right)dx + \int_{s_2}^{L}\left(\tilde{T}\tilde{T}_x + \tilde{T}_x\tilde{T}_{xxx}\right)dx.$$

We note here that the expression above contains the third spatial derivative. Since T and $\bar{T}$ are solutions to the parabolic heat equation on the time-varying domains $(0,s)\cup(s,L)$ and $(0,\bar{s})\cup(\bar{s},L)$, respectively, they will be at least three times differentiable. This can be shown using an appropriate change of variables and Theorem 3.10, p. 72 in [14]. Therefore, $\tilde{T}$ will also have the third spatial derivative except at the boundary points.

Now, integrating by parts, applying the boundary conditions from (1) and combining like terms gives

$$\dot{V}(\tilde{T},t) = -a\int_0^L\left(\tilde{T}_x^2 + \tilde{T}_{xx}^2\right)dx - a\tilde{u}(0)\left(\tilde{T}(0) + \tilde{T}_{xx}(0)\right) - a\tilde{T}_x\left(\tilde{T} + \tilde{T}_{xx}\right)\Big|_{s_1^-}^{s_1^+} - a\tilde{T}_x\left(\tilde{T} + \tilde{T}_{xx}\right)\Big|_{s_2^-}^{s_2^+} - \frac{1}{2}\dot{s}_1\tilde{T}_x^2\Big|_{s_1^-}^{s_1^+} - \frac{1}{2}\dot{s}_2\tilde{T}_x^2\Big|_{s_2^-}^{s_2^+}.$$

Hence, if the control $u(t)$ satisfies (6), then

$$\dot{V}(\tilde{T},t) = -a\int_0^L\left(\tilde{T}_x^2 + \tilde{T}_{xx}^2\right)dx =: W(\tilde{T}) \leq 0. \quad (10)$$

Now, we consider the degenerate case, in which $\bar{s}(t) = s(t)$ for some time interval of length greater than zero. This means $\tilde{T}(s(t),t) = 0$ in this interval, and since the boundaries move as governed by (2),

$$T_x(s^-) - T_x(s^+) = \bar{T}_x(s^-) - \bar{T}_x(s^+) \Rightarrow \tilde{T}_x(s^+) = \tilde{T}_x(s^-) =: \tilde{T}(s). \quad (11)$$

Then (6) simplifies to

$$u(t) = \bar{u}(t) - \frac{2\tilde{T}_x(s)}{\tilde{T}(0) + \tilde{T}_{xx}(0)}\tilde{T}_{xx}(x)\Big|_{s^-}^{s^+}.$$

Using these relationships and (3), we can again take the time derivative of (7), which in the degenerate case only has a single boundary. After integrating by parts,

$$\dot{V}(\tilde{T},t) = -a\int_0^L\left(\tilde{T}_x^2 + \tilde{T}_{xx}^2\right)dx + a\tilde{T}_x(s)\tilde{T}_{xx}(x)\Big|_{s^-}^{s^+}. \quad (12)$$

If $\tilde{T}_x(s) = 0$, then (10) clearly holds. If $\tilde{T}_x(s) > 0$, then for all $\varepsilon > 0$ sufficiently small, $T(s+\varepsilon) > 0$. If $\tilde{T}_{xx}(s^+) > 0$, then by (3), $\tilde{T}_t(s+\varepsilon) > c > 0$ for all $\varepsilon > 0$ sufficiently small. This means $\tilde{T}(s(t)+\varepsilon, t+\delta) > 0$ for all $\delta > 0$ sufficiently small. But, by assumption (A3), within the degenerate time interval,

$$s(t+\delta) = \bar{s}(t+\delta) > \bar{s}(t) = s(t) \Rightarrow s(t+\delta) = s(t) + \varepsilon$$

for some $\varepsilon > 0$. This means, taking δ small enough,

$$0 = \tilde{T}(s(t+\delta), t+\delta) = \tilde{T}(s(t)+\varepsilon, t+\delta) > 0.$$

By contradiction, then, $\tilde{T}_{xx}(s^+) \leq 0$. Similarly, $\tilde{T}_{xx}(s^-) \geq 0$. Therefore,

$$a\tilde{T}_x(s)\tilde{T}_{xx}(x)\Big|_{s^-}^{s^+} \leq 0$$

and (10) follows from (12). The same argument holds under reversed signs in the case $\tilde{T}_x(s) < 0$. Thus, in the degenerate case, under the given control law, the estimate (10) is still valid.

As an immediate conclusion of (9) and (10), under this control law the reference error $\tilde{T}$ is bounded in the $W^{1,2}(0,L)$ Sobolev norm.

We now apply an invariance principle for general evolution equations from [15]. Define the spaces $X = W^{1,2}(0,L)$ and $Y = C^0(0,L)$, and let $f(x)$ be an admissible initial value for the reference error. That is, $f(x) = T_0 - \bar{T}_0$ where T_0 and $\bar{T}_0$ satisfy assumptions (A1) and (A2). Define $G := \gamma(f) := \bigcup_{t\geq 0}\{S(t)f\}$ where $S(t)f$ is the solution to the error equations under the given control law. Since solutions to the Stefan problem are continuous and piecewise-C^2, $G \subset X$. Further, it can be shown using a slight extension of the Rellich-Kondrachov theorem in [16] that X is compactly embedded in Y. Therefore, G is compactly embedded in Y and, as noted above, G is X-bounded. Define

$$\hat{V}(y) := \int_0^L\left(\tilde{T}^2 + \tilde{T}_x^2\right)dx \text{ and } \hat{W}(y) := \int_0^L\left(\tilde{T}_x^2 + \tilde{T}_{xx}^2\right)dx$$

to be, respectively, the extensions of V and W (defined in (10)) to $Cl_Y G$, the closure of G in the supremum norm. Since functions in G will be twice differentiable almost everywhere, both of these functionals are well defined, positive semi-definite, and lower semi-continuous on $Cl_Y G$.

Thus, all the conditions of Theorem 6.3, p. 195, in [15] are met, giving the following result:

$$\lim_{t\to\infty} d_y\left(S(t)f, M_3\right)=0 \tag{13}$$

where

$$M_3 := \left\{y\in Cl_Y G : \hat{W}(y)=0\right\}.$$

In general, $M_3 := \left\{\tilde{T}(x) : \tilde{T}_x(x)\equiv 0 \equiv \tilde{T}_{xx}(x)\right\}$, that is $T(x)=\bar{T}(x)+C$ for some constant C. So, consider any constant element $\tilde{T}(x)\equiv C$ in G. If $C\neq 0$, then $s\neq\bar{s}$, but since T is continuously differentiable except at s,

$$\bar{T}_x\left(\bar{s}^+\right)=T_x\left(\bar{s}^+\right)=T_x\left(\bar{s}^-\right)=\bar{T}_x\left(\bar{s}^-\right).$$

Then by (2),

$$\dot{\bar{s}}=-b\left(\bar{T}_x\left(\bar{s}^+\right)-\bar{T}_x\left(\bar{s}^-\right)\right)=0.$$

This contradicts assumption (A3). This means that $M_3\cap G=\{0\}$, and since $M_3\subset Cl_Y G$, $M_3=\{0\}$. Therefore, (13) is equivalent to

$$\lim_{t\to\infty}\left\|\tilde{T}(x,t)\right\|_\infty=0. \quad \square$$

Remark 1. It does not follow from Theorem 1 that the solidification front position converges as well. If the temperature gradient in the reference profile is small, the solidification front position error may be arbitrarily large for small temperature errors. For practical applications, though, this gradient is not small, and the solidification front converges to the reference position as illustrated in the simulation in Section VI.

Remark 2. The well-posedness of the 1-D Stefan problem has been examined in depth, e.g. in [1, 17, 18], typically requiring boundedness of the boundary conditions and their time derivatives. The control law (6) may be unbounded, and therefore it may be necessary to regularize it in order to prove the general well-posedness of the closed-loop system. In the simulations, some regularity is attained by bounding the control, which does not result in the loss of convergence. A rigorous analysis of this issue will be carried out in subsequent work.

Remark 3. The presence of the second spatial derivative of the temperature error in the control law (6) ensures error convergence by inducing the relatively strong $W^{1,2}(0,L)$ Sobolev norm topology, but it also places additional smoothing requirements on the measurements. Relaxing the topology and removing the second spatial derivative yields a second control law given below that only depends on the first spatial derivative. However, it is only proven to be stable relative to the reference temperature, with the convergence conjectured based on given simulation results.

Theorem 2. Let the system (1)-(2) be controlled such that

$$\begin{aligned} u(t) &= \bar{u}(t)+\frac{1}{b\tilde{T}(0)}\left[\dot{s}\tilde{T}(s)-\dot{\bar{s}}\tilde{T}(\bar{s})\right] \\ &= \bar{u}(t)-\frac{1}{\tilde{T}(0)}\left[\tilde{T}(s)\tilde{T}_x(x)\Big|_{s^-}^{s^+}-\tilde{T}(\bar{s})\tilde{T}_x(x)\Big|_{\bar{s}^-}^{\bar{s}^+}\right] \end{aligned} \tag{14}$$

where the initial conditions satisfy (A1) and (A2). Then, the reference error $\tilde{T}(x,t)$ is bounded in the L^2 norm.

Proof: Consider the Lyapunov functional

$$\begin{aligned} V\left(\tilde{T}\right) &:= \frac{1}{2}\int_0^L \tilde{T}^2 dx = \frac{1}{2}\left\|\tilde{T}\right\|_2^2 \\ &= \frac{1}{2}\int_0^{s_1}\tilde{T}^2 dx+\frac{1}{2}\int_{s_1}^{s_2}\tilde{T}^2 dx+\frac{1}{2}\int_{s_2}^{L}\tilde{T}^2 dx, \end{aligned} \tag{15}$$

where $s_1 := \min\{s,\bar{s}\}$ and $s_2 := \max\{s,\bar{s}\}$. As in Theorem 1, we take the time derivative and integrate by parts, substituting in the PDEs and boundary conditions where appropriate. The result is

$$\dot{V}\left(\tilde{T},t\right)=-a\left[\int_0^L \tilde{T}_x^2 dx-\tilde{u}(t)\tilde{T}(0)+\frac{1}{b}\dot{s}\tilde{T}(s)-\frac{1}{b}\dot{\bar{s}}\tilde{T}(\bar{s})\right].$$

If the control law satisfies (14),

$$\dot{V}\left(\tilde{T},t\right)=-a\int_0^L \tilde{T}_x^2 dx\leq 0. \tag{16}$$

In the degenerate case $s=\bar{s}$, control law (14) reduces to $u=\bar{u}$. Again taking the time derivative and integrating by parts gives (16), where the boundary terms drop out because $\tilde{u}=0$ and $\tilde{T}(s)=\tilde{T}(\bar{s})=0$. Therefore, $V\left(\tilde{T}\right)$, and consequently $\left\|\tilde{T}\right\|_2$, is bounded over time. □

Although the proof does not guarantee convergence, the control law in simulation has shown convergent behavior. Therefore, we formulate the following conjecture.

Conjecture 1. Let the system (1)-(2) be controlled such that *u*(*t*) is given by (14) where the initial conditions satisfy (A1) and (A2) and the reference temperature history satisfies (A3). Then, the reference error $\tilde{T}(x,t)$ converges in the L^p norm, $p\leq 2$, to an ε-neighborhood of zero reference error.

A plausible proof could be based on the results in [15] as in Theorem 1, or use Barbalat's Lemma. Either method would require showing that $\left\|\tilde{T}_x\right\|_2$ is bounded along trajectories of the error system under control (14).

IV. Applicability of the Control Law

Assumptions (A1) and (A2) will be true for all physically possible initial conditions. Assumption (A3) is generally true for any practical reference profile. An alternative to (A3) ensuring convergence to the reference system is:

(A4) The initial conditions satisfy $T_0(x)=T_f$ for $x\geq s_0$, and $\bar{T}_0(x)=T_f$ for $x\geq\bar{s}_0$.

Under this assumption, from the boundary conditions at $x=s(t)$ and $x=L$, it follows that $\tilde{T}(x)=0$ for $x\geq s_2$ and all $t\geq 0$. This means $M_3\cap G=\{0\}$ in the proof of Theorem 1, and the conclusion still holds.

There are two ways in which the model given by (1)-(2) significantly differs from the physical system. First, we assume boundary heat flux, when in fact control is limited to the cooling water sprays that have fixed, spatially varying footprints. Moreover, the water flow rates are strictly limited by the spray piping system. Although Section III does not investigate saturation, we placed bounds on the control signals in the simulations discussed in Section IV and conjecture that the controlled system converges for initial conditions close to zero reference error.

Second, we have assumed full state feedback is available. It is clear that in the real process the temperature at any point below the surface cannot be measured. An important area for future improvement of this work, then, is in output feedback design, which is briefly addressed in the next section.

V. Dirichlet Control and Estimator Design

First, we consider a controller in which the boundary surface temperature can be set exactly equal to the reference.

Theorem 3. Let the reference and the actual temperatures satisfy assumptions (A1), (A2), and (A4). In addition, assume that $T_0(x) < T_f$ for $x < s_0$ and $\bar{T}_0(x) < T_f$ for $x < \bar{s}_0$, and that $\bar{T}(0,t) < T_f$ for all time. If the system (1)-(2) is controlled such that

$$T(0,t) = \bar{T}(0,t) \tag{17}$$

for all time, then the reference error $\tilde{T}(x,t)$ is bounded in the L^2 norm.

Proof: Under these assumptions, applying the maximum principle for parabolic equations, $T(x,t) < T_f$ for all $x < s(t)$. This means that $T_x(s^-) < 0$, and noting the signs in (2), $\dot{s} > 0$ for all time. The same holds for $\bar{T}$ and $\bar{s}$.

Again we use (15) as a Lyapunov functional candidate, and take the time derivative. Integrating by parts and applying the boundary conditions,

$$\dot{V}(\tilde{T},t) = -a\left[\int_0^L \tilde{T}_x^2 dx + \frac{1}{b}\dot{s}\tilde{T}(s) - \frac{1}{b}\dot{\bar{s}}\tilde{T}(\bar{s})\right].$$

Under assumption (A4), if $s > \bar{s}$, then $T(s) = \bar{T}(s) = T_f$, and from above, $T(\bar{s}) < T_f = \bar{T}(s)$. This means $\dot{s}\tilde{T}(s) = 0$ and $\dot{\bar{s}}\tilde{T}(\bar{s}) < 0$. Similarly, if $s < \bar{s}$, then $\dot{s}\tilde{T}(s) > 0$ and $\dot{\bar{s}}\tilde{T}(\bar{s}) = 0$. In the degenerate case $s = \bar{s}$, the boundary terms drop out because $\tilde{T}(s) = \tilde{T}(\bar{s}) = 0$. In either case,

$$\dot{V}(\tilde{T},t) \le -a\int_0^L \tilde{T}_x^2 dx \le 0,$$

and thus $\|\tilde{T}\|_2$, is bounded over time. □

Although not proved, it seems reasonable to conjecture as with Theorem 2 that there is some convergence. Also, since the surface temperature is strongly affected by the cooling water sprays, the Dirichlet boundary condition (17) can often be achieved in practice. The strength of this result is that it only requires knowledge of $T(0,t)$, which can realistically be measured. Theorem 3 also immediately gives a possible estimator design.

TABLE I

Thermodynamic Properties Used in Simulations

Symbol	Description	Value
a	thermal diffusivity	3.98 x 10^{-6} W/m·K
b	Stefan condition constant	1.102 x 10^{-8} m^2/K·s
T_f	melting temperature	1783 K
L	half-thickness of strand	0.1 m

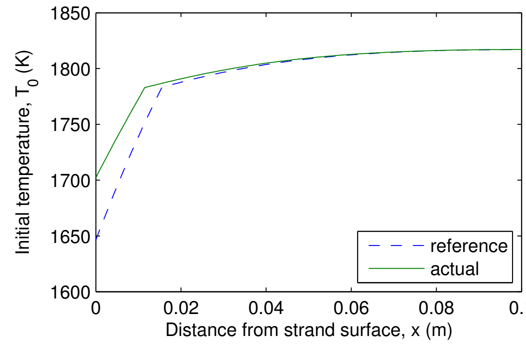


Fig. 1. Initial temperature profiles for reference temperature $\bar{T}_0$ and actual temperature T_0.

Corollary 1. Define the feedback-based estimates $\hat{T}(x,t)$ and $\hat{s}(t)$ to be a solution to (1)-(2) with the Dirchlet boundary condition, based on boundary measurement of the plant, $\hat{T}(0,t) = T(0,t)$. Then, if T and $\hat{T}$ satisfy assumptions (A1), (A2), and (A4), and for all times t, $T(0,t) < T_f$, the estimation error is bounded in the L^2 norm.

This leads us to the following conjecture for an output-feedback controller design.

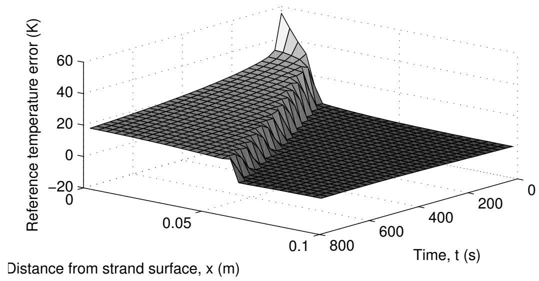
Conjecture 2. Let $\hat{T}(x,t)$ and $\hat{s}(t)$ be the estimates of the plant $T(x,t)$ and $s(t)$ using the output injection described in Corollary 1. Let the plant be controlled using the certainty equivalence method, i.e. calculating control law (6) or (14) based on the estimates. Then the reference error $\tilde{T}(x,t)$ converges to an ε-neighborhood of zero reference error in the L^2 norm.

Although this conjecture is unproven, it is supported by simulation results.

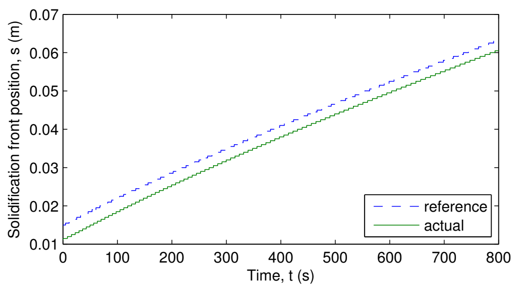
VI. Simulation Results

The following simulation results use the parameters in Table 1, based on ULC (ultra-low carbon) steel. The initial conditions are shown in Figure 1. The simulations employ an enthalpy-based method to model solidification, rather than an actual moving boundary. The simulation code was verified against an analytical solution to the Stefan problem from [19]. The controlled simulations were found to be very noisy, as seen in Figure 3c, and bounds were put on the control values as discussed in Section IV.

Figure 2 shows the behavior of the system under open-loop control with $u(t) = \bar{u}(t)$ for all $t \ge 0$. In this case, the reference errors in both temperature and solidification front



a) Reference temperature error $\tilde{T}(x,t)$



b) Solidification front positions $\bar{s}(t)$ and $s(t)$

Fig. 2. Simulation results for system (1)-(2) with no control action.

position appear to converge to constant, non-zero values. This approximates the current spray cooling state-of-the-art in most continuous casters, in which spray practices do not account for changes in superheat or mold heat removal.

Figure 3 shows simulation results using control law (6). Although not shown here, results were similar using control law (14) and the output-feedback control method described in Conjecture 2. Although convergence was not proved for bounded control values, the reference temperature error appears to converge to 0.

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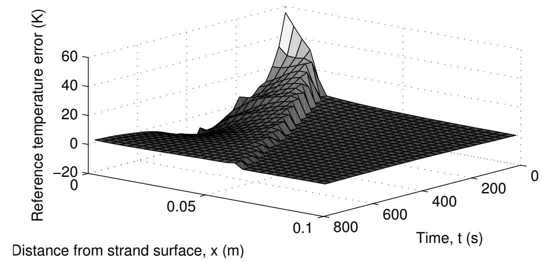
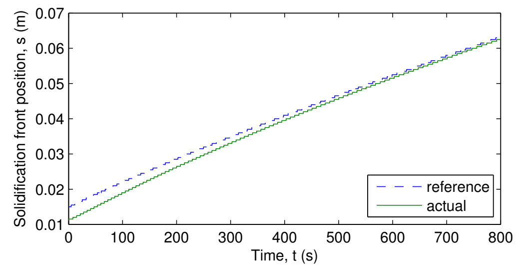
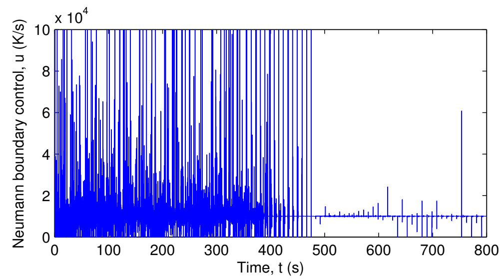


a) Reference temperature error $\tilde{T}(x,t)$



b) Solidification front positions $\bar{s}(t)$ and $s(t)$



c) Neumann boundary control $u(t)$

Fig. 3. Simulation results for system (1)-(2) under control law (6).

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